Plancherel-Rotach Asymptotics of Second-Order Difference Equations with Linear Coefficients

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Introduction

All of the classical hypergeometric (monic) orthogonal polynomials $\pi_n(x)$ within Askey scheme satisfy the following second-order linear difference equation.

$$\pi_{n+1}(x) = (x - A_n)\pi_n(x) - B_n\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x - A_0,$$

where the coefficients $A_n$ and $B_n$ are polynomials or rational functions of $n$.

- Charlier polynomials: $A_n = n + a$ and $B_n = na$.
- Hermite polynomials: $A_n = 0$ and $B_n = n/2$.
- Chebyshev polynomials: $A_n = 0$ and $B_n = 1/4$. 
An illustrative example

We consider the following difference equation:

$$\pi_{n+1}(x) = (x - n)\pi_n(x) - n\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x.$$ 

The first few polynomials:

$$\pi_2(x) = x^2 - x + 1;$$
$$\pi_3(x) = x^3 - 3x^2 + 5x - 2;$$
$$\pi_4(x) = x^4 - 6x^3 + 17x^2 - 20x + 9;$$
$$\pi_5(x) = x^5 - 10x^4 + 45x^3 - 100x^2 + 109x - 44;$$
$$\pi_6(x) = x^6 - 15x^5 + 100x^4 - 355x^3 + 694x^2 - 689x + 265;$$

\vdots
Zeros of $\pi_n(\sqrt{n}z + n)$
Questions

What is the Y-shape curve? What is the intersection point?
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Answer. The V-shape part can be determined by the following equation:

\[
\text{Re} \left\{ 2\sqrt{z - 2i\sqrt{z + 2i}} - z \ln \frac{z + \sqrt{z - 2i\sqrt{z + 2i}}}{z - \sqrt{z - 2i\sqrt{z + 2i}}} \right\} = 0.
\]

The intersection point is the real root of the following equation:

\[
2\sqrt{z^2 + 4} - z \ln \frac{z + \sqrt{z^2 + 4}}{-z + \sqrt{z^2 + 4}} = 0.
\]

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n + c}{2}\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x.$$ 


$$\pi_{n+1}(x) = (x - n - a - c)\pi_n(x) - a(n + c)\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x - a - c.$$ 

Question: what are the large-$n$ behaviors of the associated polynomials?
Second-Order Difference Equations with Linear Coefficients

Let $A_n$ and $B_n$ be linear functions in $n$. Upon a shift on $x$, we set $A_n = dn$ and $B_n = an + b$.

$$
\pi_{n+1}(x) = (x - dn)\pi_n(x) - (an + b)\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x.
$$

- Case I: $d \neq 0$. By symmetry, we may assume $d > 0$.
  - Case I.A: $d > 0$ and $a > 0$;
  - Case I.B: $d > 0$ and $a < 0$;
  - Case I.C: $d > 0$ and $a = 0$.

- Case II: $d = 0$.
  - Case I.A: $d = 0$ and $a > 0$;
  - Case I.B: $d = 0$ and $a < 0$;
  - Case I.C: $d = 0$ and $a = 0$.
Main idea I: ratio asymptotics

Consider case I.A \((d > 0 \text{ and } a > 0)\), we define

\[
w_k(x) := \frac{\pi_k(x)}{\pi_{k-1}(x)}
\]

for any \(k \geq 1\). It follows that

\[
w_{k+1}(x) = x - dk - \frac{ak + b}{w_k(x)}, \quad k \geq 1.
\]

Let \(x = ny\) with \(y \in \mathbb{C} \setminus [0, d + 2\sqrt{a}/\sqrt{n}]\). We can derive asymptotic formulas for \(w_k(x)\) for all \(k = 1, 2, \cdots, n\) by successive approximation and induction.
Main idea I: ratio asymptotics

We have as $n \to \infty$,

$$w_k(x) \sim \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2}$$

$$\quad \times \left\{ 1 + \frac{d}{2\sqrt{(x - dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x - dk)^2 - 4ak]} \right\}.$$

Therefore,

$$\ln \pi_n(x) \sim \sum_{k=1}^{n} \left\{ \ln \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2}$$

$$\quad + \frac{d}{2\sqrt{(x - dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x - dk)^2 - 4ak]} \right\}.$$
Main idea II: matching principle

A detailed calculation yields

\[
\pi_n(nd + \sqrt{n}z) \sim (n/e)^n \left( \frac{\sqrt{a}}{\sqrt{n}} \right)^{-a/d^2} \sqrt{n}z/d \left( d + \frac{z}{\sqrt{n}} \right)^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \\
\times \left( \frac{\sqrt{nd} + z}{\sqrt{4a - z^2}} \right)^{1/2} \exp \left( \frac{2a - z^2 - 4\sqrt{n}dz}{4d^2} \right) \\
\times 2 \cos \left[ \left( -\frac{a}{d^2} - \sqrt{n}z/d \right) \arccos \frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z + 4\sqrt{nd})\sqrt{4a - z^2}}{4d^2} \right]
\]

for \( z \in \mathbb{C} \setminus [-\sqrt{nd}, 2\sqrt{a}] \).

A matching principle gives asymptotic formula for \( z \) in a complex neighborhood of any compact subset in \(( -\sqrt{nd}, -2\sqrt{a})\).
Main idea II: matching principle

Assume that
\[ \pi_n(z) \sim \Phi(n, z), \quad z \in \mathbb{C} \setminus \overline{S}_0, \]
where \( S_0 \) is the union of one-dimensional (open) branch cuts/Stokes lines of \( \Phi(n, z) \) with \( \overline{S}_0 \) being its closure. We can obtain the asymptotic formula of \( \pi_n(z) \) on \( S_0 \) as follows. Denote by \( \Phi^\pm(n, x) \) the one-sided limits of \( \Phi(n, z) \) on \( S_0 \). If \( \Phi^\pm(n, x) \) can be analytically extended in a neighborhood of \( S_0 \) and \( \Phi^+(n, z) \) dominates on the + side of \( S_0 \), whereas \( \Phi^-(n, z) \) dominates on the – side of \( S_0 \), we then have
\[ \pi_n(x) \sim \Phi^+(n, x) + \Phi^-(n, x), \quad x \in S_0. \]
Especially, the curve \( S_0 \) can be determined by solving the equation
\[ \lim_{n \to \infty} \frac{\ln |\Phi^+(n, z)|}{\ln |\Phi^-(n, z)|} = 1. \]
Main idea II: matching principle

For \( z \) in a complex neighborhood of any compact subset in \((-2\sqrt{a}, 2\sqrt{a})\),

\[
\pi_n(nd + \sqrt{n}z) \sim (n/e)^n (\frac{\sqrt{a}}{\sqrt{n}})^{-a/d^2 - \sqrt{n}z/d} (d + \frac{z}{\sqrt{n}})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left( \frac{\sqrt{nd} + z}{\sqrt{4a - z^2}} \right)^{1/2} \\
\times e^{\frac{2a-z^2-4\sqrt{nd}z}{4d^2}} \times 2 \cos \left[ \left( -\frac{a}{d^2} - \frac{\sqrt{n}z}{d} \right) \arccos \frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{\left( z + 4\sqrt{nd}\right)\sqrt{4a - z^2}}{4d^2} \right].
\]

For \( z \) in a complex neighborhood of any compact subset in \((-\sqrt{nd}, -2\sqrt{a})\),

\[
\pi_n(nd + \sqrt{n}z) \sim (n/e)^n (\frac{-z + \sqrt{-z - 2\sqrt{a}\sqrt{-z + 2\sqrt{a}}}}{2\sqrt{n}})^{-a/d^2 - \sqrt{n}z/d} \\
\times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left( \frac{\sqrt{nd} + z}{\sqrt{-z - 2\sqrt{a}\sqrt{-z + 2\sqrt{a}}} \sqrt{4a - z^2}} \right)^{1/2} \\
\times e^{\frac{2a-z^2-4\sqrt{nd}z-(z+4\sqrt{nd})\sqrt{-z-2\sqrt{a}\sqrt{-z+2\sqrt{a}}}}{4d^2}} \times 2 \cos \left[ \pi(-a/d^2 - \sqrt{n}z/d - 1/2) \right].
\]
Associated Hermite polynomials


\[ \pi_{n+1}(x) = x\pi_n(x) - \frac{n + c}{2}\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x. \]

These polynomials belong to the case II.A with \( d = 0, \ a = 1/2 \) and \( b = c/2 \).

Let \( x = \sqrt{n}y \). As \( n \to \infty \), we have for \( y \in \mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}] \),

\[
\pi_n(\sqrt{n}y) \sim \left(\frac{n}{4e}\right)^{n/2}(y + \sqrt{y^2 - 2})^n \left(\frac{y + \sqrt{y^2 - 2}}{2\sqrt{y^2 - 2}}\right)^{1/2} \left(\frac{y + \sqrt{y^2 - 2}}{2y}\right)^c \\
\times \exp\left[\frac{ny}{2}(y - \sqrt{y^2 - 2})\right].
\]
Associated Hermite polynomials

For $y$ in a complex neighborhood of any compact subset in $(0, \sqrt{2})$, we have

$$\pi_n(\sqrt{ny}) \sim \left(\frac{n}{2e}\right)^{n/2} \left(\frac{\sqrt{1/2}}{\sqrt{2 - y\sqrt{2 + y}}}\right)^{1/2} \left(\frac{\sqrt{1/2}}{y}\right)^c \times \exp\left(\frac{ny^2}{2}\right)$$

$$\times 2 \cos\left[(n + 1/2 + c) \arccos \frac{y}{\sqrt{2}} - \pi/4 - \frac{ny}{2} \sqrt{2 - y\sqrt{2 + y}}\right].$$

For $y$ in a complex neighborhood of any compact subset in $(-\sqrt{2}, 0)$, we have

$$\pi_n(\sqrt{ny}) \sim \left(\frac{n}{2e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{1/2}}{\sqrt{2 - y\sqrt{2 + y}}}\right)^{1/2} \left(\frac{\sqrt{1/2}}{-y}\right)^c \times \exp\left(\frac{ny^2}{2}\right)$$

$$\times 2 \cos\left[(n + 1/2 + c) \arccos \frac{-y}{\sqrt{2}} - \pi/4 + \frac{ny}{2} \sqrt{2 - y\sqrt{2 + y}}\right].$$
Associated Charlier polynomials


$$
\pi_{n+1}(x) = (x - n - a - c)\pi_n(x) - a(n + c)\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x - a - c.
$$

These polynomials belong to the case I.A with $d = 1$ and $b = ac$.

Let $x = a + c + ny$ and $y = 1 + z/\sqrt{n}$. As $n \to \infty$, for $z \in \mathbb{C} \setminus [-\sqrt{n}, 2\sqrt{a}]$, we have

$$
\pi_n(n + \sqrt{n}z + a + c) \sim (n/e)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}}\right)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{n} + z)}\right)^{-a - \sqrt{n}(\sqrt{n} + z)}
	imes (\frac{\sqrt{n} + z}{\sqrt{z^2 - 4a}})^{1/2} \times \exp\left[\frac{2a - z^2 - 4\sqrt{n}z + (z + 4\sqrt{n})\sqrt{z^2 - 4a}}{4}\right].
$$
Associated Charlier polynomials

For \( z \) in a complex neighborhood of a compact subset in \((-2\sqrt{a}, 2\sqrt{a})\), we have

\[
\pi_n(n + \sqrt{n}z + a + c) \sim (n/e)^n \left( \frac{\sqrt{4a - z^2}}{\sqrt{n}} \right)^{-a - \sqrt{n}z} (1 + z/\sqrt{n})^a + \sqrt{n}(\sqrt{n} + z) \left( \frac{\sqrt{n} + z}{\sqrt{4a - z^2}} \right)^{1/2}
\]

\[
\times e^{(2a - z^2 - 4\sqrt{n}z)/4} \times 2 \cos \left[ (-a - \sqrt{n}z) \arccos \frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z + 4\sqrt{n})\sqrt{4a - z^2}}{4} \right].
\]

For \( z \) in a complex neighborhood of a compact subset in \((-\sqrt{n}, -2\sqrt{a})\), we have

\[
\pi_n(n + \sqrt{n}z + a + c) \sim \left( \frac{-z + \sqrt{-z - 2\sqrt{a}\sqrt{-z + 2\sqrt{a}}}}{2\sqrt{n}} \right)^{-a - \sqrt{n}z}
\]

\[
\times e^{\frac{2a - z^2 - 4\sqrt{n}z - (z + 4\sqrt{n})\sqrt{-z - 2\sqrt{a}\sqrt{-z + 2\sqrt{a}}}}{4}} (1 + z/\sqrt{n})^a + \sqrt{n}(\sqrt{n} + z)
\]

\[
\times (n/e)^n \left( \frac{\sqrt{n} + z}{\sqrt{-z - 2\sqrt{a}\sqrt{-z + 2\sqrt{a}}}} \right)^{1/2} \times 2 \cos \left[ \pi(-a - \sqrt{n}z - 1/2) \right].
\]
Future work

- Uniform asymptotic analysis at the origin for cases II.A and II.B, namely, $d = 0$ and $a \neq 0$.

- Uniform asymptotic analysis at the intersection point of the Y-shape curve for case I.B ($d > 0$ and $a < 0$).

- The coefficients $A_n$ and $B_n$ are quadratic functions, polynomials, rational functions, or exponential functions in $n$.

- Higher-order linear difference equations.
Thank you!